DETERMINATION OF THE SHAPE OF A BODY OF MINIMUM DRAG AT HYPERSONIC SPEED

(OPREDELENIE FORMY TEL MINIMAL'NOGO SOPROTIVLENIIA PRI BOL'SHIKH SVERKHZVUKOVYKH SKOROSTIAKH)

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In recent years several variational problems for the shape of a body for minimum drag at hypersonic flight speeds have been solved on the basis of Newton's law of resistance [1].

Solution of the variational problem in a more exact formulation, using A. Busemann's law of resistance, was proposed in [2,3,5]. However, as was shown by Hayes [3], in the improved formulation the contour of the body of minimum drag should have a discontinuity in slope at the end point, because then, according to Busemann's law, an infinite negative pressure appears at that point, reducing the drag by a finite amount. The physical pressure cannot be negative, and a change in slope at the end-point should not, in supersonic flow, influence the pressure distribution upstream and hence the drag.

This disagreement with the physics of supersonic flow requires a new formulation of the variational problem with revised conditions, so that the pressure on the body contour will be everywhere nonnegative.

A general method is given below for the solution of this problem for plane and axisymmetric gas flow.

We consider flow past a body with, in general, a duct, in a plane or axisymmetric hypersonic gas stream. Assuming that all characteristic dimensions are referred to the length of the body, we take its length as unity.

The drag coefficient of the body according to Busemann's formula is then, after a transformation given in [2], equal to

$$C_x = \frac{2\nu}{r_1^{\nu} - r_0^{\nu}}\left[\frac{r_1^{\nu} - r_0^{\nu}}{\nu} - \frac{1}{(1 + r_1'^2)^{1/2}}\int_0^1 \frac{r^{\nu-1} r' dz}{(1 + r'^2)^{1/2}}\right] \qquad (1)$$

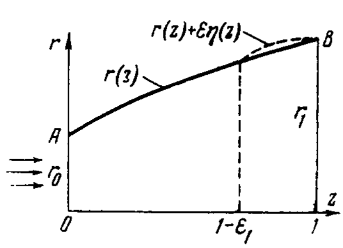

Fig. 1.

The notation is given in Fig. 1; the quantity $\nu = 1$ for plane flow and $\nu = 2$ for axisymmetric flow, and r_1' is the derivative at the end-point of the contour.

In calculating the coefficient C_x for a body of revolution, the drag force is referred to the annular area $\pi(r_1^2 - r_0^2)$; in the case of plane flow $r_0 = 0$, and the drag force acting on only one side of the profile is considered. The contour is assumed to be smooth and to have only finite discontinuities in second derivative at discrete points. Under these assumptions it is valid to write Busemann's formula in the form (1).

For given values of the quantities r_0 and r_1 it follows from (1) that the drag of the body attains an absolute minimum for the maximum value of the integral and the condition that $r_1' = 0$. The corresponding contour was found in the papers mentioned above and, as was remarked, is not smooth.

Writing the condition that the pressure on the contour be nonnegative we have, according to [2]

$$r'^2 + \frac{r''}{r^{\nu-1}(1 + r'^2)^{1/2}}\int_0^z \frac{r' r^{\nu-1}\, dz}{(1 + r'^2)^{1/2}} \geqslant 0 \qquad (2)$$

We will solve the variational problem for a body of given lengths (r_0 and r_1) under condition (2), after first proving the following theorem:

Theorem. The minimum of Expression (1) for a body of given length under condition (2) can be attained only on a curve along which, in a certain finite part $c \leqslant z \leqslant 1$, the inequality (2) becomes an equality, that is, the pressure vanishes.

Proof. Assuming the contrary, let the minimum of (1) be attained on a certain curve $r(z)$ along which the pressure is everywhere greater than zero. We then consider Expression (1) on some neighboring curve $r^{\circ}(z) = r(z) + \epsilon\,\eta(z)$, where $\eta(z)$ is different from zero only in the interval $1 - \epsilon_1 < z < 1$ and possesses necessary smoothness properties (ϵ and ϵ_1 are small quantities). If the inequality (2) is satisfied strictly on the curve $r(z)$, then for sufficiently small ϵ it will be satisfied also for the function $r^{\circ}(z)$, since the left-hand side of (2) varies as a quantity of $O(\epsilon)$. Substituting $r^{\circ}(z)$ into the functional (1), we find that the second term in square brackets becomes, after a simple calculation

$$\frac{1}{(1+r_1'^2)^{1/2}}\int_0^1 \frac{r^{\nu-1}r'\,dz}{(1+r'^2)^{1/2}} - \frac{\varepsilon\eta'(1)}{(1+r_1'^2)^{3/2}}\int_0^1 \frac{r^{\nu-1}r'\,dz}{(1+r'^2)^{1/2}} + O(\varepsilon\varepsilon_1)$$

Now, choosing the function $\eta(z)$ with the value of its derivative $\eta'(1) < 0$ (which is always possible, because it is easy to show that $r'(1) > 0$), we obtain from (1) that the value of C_x on the curve $r^{\circ}(z)$ is less than on the curve $r(z)$. Thus, we arrive at a contradiction and, consequently, the desired extremal has a finite piece $c \leqslant z \leqslant 1$ with zero pressure. All other possibilities are ruled out except the case $c = 1$, which we now consider.

In fact, if we put $c = 1$, the equality (2) is satisfied identically at this point for all values $r_c' > 0$ (cf. also (4)). Consequently, it is possible to repeat in full the proof of the theorem and arrive at a contradiction. Here the inequality (2) can be satisfied on the comparison line if, for example, we set $r^{\circ\prime\prime} = 0$ in the interval $1 - 1/2\,\epsilon_1 < z < 1$, and at the point $z = 1 - 1/2\,\epsilon_1$ introduce a finite discontinuity in the function $\eta''(z)$ such that the function $\eta''(z)$ is of order unity on the segment $1 - \epsilon_1 \leqslant z \leqslant 1 - 1/2\,\epsilon_1$.

We proceed to the investigation of the manifold of curves of zero pressure. On this manifold the inequality (2) becomes an equality, and it is not difficult to see that it represents the derivative of the expression

$$\frac{r'}{(1+r'^2)^{1/2}}\int_0^z \frac{r'r^{\nu-1}\,dz}{(1+r'^2)^{1/2}} = \text{const} \qquad (3)$$

Hence, in particular, it follows that it is impossible to pass the contour of zero pressure through the initial point $z = 0$.

Eliminating the integral from (2) and (3) we obtain an ordinary differential equation of second order for the function $r(z)$. The general solution, representing the family of curves of zero pressure on the segment $c \leqslant z \leqslant 1$, has the form*

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* In other variables the curve of zero pressure was first obtained by Lighthill [4].

$$\frac{r_1^{\nu+1} - r^{\nu+1}}{\nu(\nu+1)} - r_c^{\nu}\left(\frac{1}{\nu} - \frac{k}{r_c'}\right)(r_1 - r) = k(1-z)$$

$$k = \frac{r_c'}{r_c^{\nu}(1+r_c'^2)^{1/2}} \int_0^c \frac{r' r^{\nu-1}\,dz}{(1+r'^2)^{1/2}} \qquad (4)$$

The constants of integration are found successively from the conditions that the curve has, at the point $C(c,\ r_c)$, the direction of the tangent r_c' and passes through the point $B(1,\ r_1)$. The form of the line of zero pressure depends, through the constant k, on the form of the stream line ahead. Investigation of Equation (4) shows that if the pressure is negative on the contour in the neighborhood of the final point, then its final portion may be replaced by a line of zero pressure tangent at the juncture. On the other hand, a contour on which the pressure is everywhere positive can, in the neighborhood of the end-point, be corrected so that the pressure at the end is negative. Hence, if in the correction of the contour a section is replaced by a line of zero pressure, then we obtain a new contour whose drag is either less than the drag of the contour with positive pressure or differs by an arbitrarily small amount. Consequently, we arrive at the conclusion that if there exists a contour of minimum drag in the class of curves with a section of zero pressure at the end, and $p \geqslant 0$ everywhere on this contour, then this minimum is the least value of the drag (1) under the condition (2).

As a result of the theorem proved, the drag of the optimum body arises only from its forward portion $0 \leqslant z \leqslant c$, and Equation (1) can be replaced by the expression

$$C_x = \frac{2\nu}{r_1^{\nu} - r_0^{\nu}}\left[\frac{r_c^{\nu} - r_0^{\nu}}{\nu} - \frac{1}{(1+r_c'^2)^{1/2}} \int_0^c \frac{r^{\nu-1} r'\,dz}{(1+r'^2)^{1/2}}\right] \qquad (5)$$

We calculate the variation of C_x considering that the quantities c, r_c, r_c' are variable. After an easy transformation we obtain

$$\delta C_x = \frac{2\nu r_c^{\nu-1} r_c'^2}{(r_1^{\nu} - r_0^{\nu})(1+r_c'^2)^2}\Big\{(2+r_c'^2)\,\delta r_c - r_c'\delta c + \qquad (6)$$

$$+ \frac{(1+r_c'^2)^{1/2}\,\delta r_c'}{r_c' r_c^{\nu-1}} \int_0^c \frac{r^{\nu-1} r'\,dz}{(1+r'^2)^{1/2}} - \frac{(1+r_c'^2)^{3/2}}{r_c'^2\, r_c^{\nu-1}} \int_0^c \left[F_r - \frac{d}{dz}(F_{r'})\right]\delta r\,dr\Big\}$$

$$F = \frac{r^{\nu-1} r'}{(1+r'^2)^{1/2}}, \qquad \delta r_c \neq \delta r \quad \text{for } z = c$$

On the optimal curve ($0 \leqslant z \leqslant c$) the variation should vanish. In this connection, if we consider the very narrow class of admissible lines passing through the end with a constant angular coefficient r_c' equal to the angular coefficient of the extremal integral standing in curly brackets (6), we conclude that in this class also the unknown curve is optimal. Consequently, according to (6) the Euler equation for the function F should be satisfied. Its solution was studied in [2,5]. In [5] it was shown that along the extremal the pressure is greater than zero.

It is convenient to carry out the investigation separately for plane and axisymmetric flows. Axisymmetric flow offers the greater difficulty, although the line of reasoning is identical in the two cases. Because the results of the solution for plane flow agree with the results obtained for this case by the non-rigorous solution of Hayes [3], we give the solution here only for axisymmetric flow.

In parametric form the extremals of the above-mentioned Euler equation [2] are determined by the equations

$$\frac{r}{c} = \frac{R(\alpha)}{z(\alpha_1) - z(\alpha_0)}, \qquad \frac{z}{c} = \frac{z(\alpha) - z(\alpha_0)}{z(\alpha_1) - z(\alpha_0)} \tag{7}$$

$$R(\alpha) = \frac{1}{\sin^3\alpha}, \qquad z(\alpha) = \frac{3}{4}\frac{\cos^3\alpha}{\sin^4\alpha} + \frac{3}{8}\frac{\cos\alpha}{\sin^2\alpha} + \frac{3}{8}\ln\tan\frac{\alpha}{2}$$

Here, the parameter $a = \tan^{-1}(dr/dz)$ varies within the limits $a_0 \leqslant a \leqslant a_1$ for $0 \leqslant z \leqslant c$.

At the end-point of the contour the following relation holds:

$$\frac{r_0}{c} = \frac{R(\alpha_0)}{z(\alpha_1) - z(\alpha_0)}, \qquad \frac{r_c}{c} = \frac{R(\alpha_1)}{z(\alpha_1) - z(\alpha_0)}, \qquad \alpha_0 = \sin^{-1}\left[\left(\frac{r_c}{r_0}\right)^{1/3}\sin\alpha_1\right] \tag{8}$$

The nose part of the optimal curve is found, as was shown, among the extremals of the Euler equation; it is therefore possible to seek it in the very narrow class of admissible lines, keeping only one of the extremals.

As a result we obtain

$$\delta c = \frac{1}{R(\alpha_1)}\left[z(\alpha_1) - z(\alpha_0) - \frac{z'(\alpha_0)}{3\cot\alpha_0}\right]\delta r_c + \frac{r_c}{R(\alpha_1)(1 + r_c'^2)}\left\{z'(\alpha_1) - \right.$$

$$\left. - \frac{R'(\alpha_1)}{R(\alpha_1)}[z(\alpha_1) - z(\alpha_0)] - \frac{z'(\alpha_0)}{r_c'\cot\alpha_0}\right\}\delta r_c' \tag{9}$$

Eliminating the quantities c and δc from Expression (6) with the aid of (8) and (9), we have the relation

$$\delta C_x = M_1(r_c, r_c')\frac{\delta r_c}{r_c} + M_2(r_c, r_c')\frac{\delta r_c'}{1+r_c'^2} = 0$$

$$M_1(r_c, r_c') = 2 + r_c'^2 - \frac{r_c'}{R(\alpha_1)}\left[z(\alpha_1) - z(\alpha_0) - \frac{z'(\alpha_0)}{3\cot\alpha_0}\right] \tag{10}$$

$$M_2(r_c, r_c') = -\frac{k(1+r_c'^2)^2}{r_c'^2} - \frac{r_c'}{R(\alpha_1)}\left\{z'(\alpha_1) - \frac{R'(\alpha_1)}{R(\alpha_1)}\left[z(\alpha_1) - z(\alpha_0) - \frac{z'(\alpha_0)}{r_c'\cot\alpha_0}\right]\right\}$$

The problem is now reduced to finding the juncture-point $C(c, r_c)$ of the extremal (7) and the line of zero pressure (4).

We put Equation (4) at point C in the form

$$\Delta^3 + 3\Delta^2 + 6k\left(\frac{1}{r_c'} - \frac{1}{r_1}\right)\Delta + 6k\left[\frac{z(\alpha_1) - z(\alpha_0)}{R(\alpha_1)} - \frac{1}{r_1}\right] = 0$$

$$\Delta = \frac{r_1}{r_c} - 1 \tag{11}$$

$$k = \frac{\sin^7\alpha_1}{2}\left[\frac{\cos\alpha}{\sin^6\alpha} - \frac{1}{4}\frac{\cos\alpha}{\sin^4\alpha} - \frac{3}{8}\frac{\cos\alpha}{\sin^2\alpha} + \frac{3}{8}\ln\tan\frac{\alpha}{2}\right]_{\alpha_0}^{\alpha_1}$$

Taking the variation of (11), we find that

$$N_1(r_c, r_c')\frac{\delta r_c}{r_c} + N_2(r_c, r_c')\frac{\delta r_c'}{1+r_c'^2} = 0 \tag{12}$$

$$N_1(r_c, r_c') = \frac{kz'(\alpha_0)}{3R(\alpha_1)\cot\alpha_0} + \frac{r_1}{r_c}\left[\frac{\Delta^2}{2} + \Delta + k\left(\frac{1}{r_c'} - \frac{1}{r_1}\right)\right] - $$

$$- \frac{k_{\alpha_0}'\tan\alpha_0}{3r_c'}\left[\Delta\left(\frac{1}{r_c'} - \frac{1}{r_1}\right) + \frac{z(\alpha_1) - z(\alpha_0)}{R(\alpha_1)} - \frac{1}{r_1}\right]$$

$$N_2(r_c, r_c') = \frac{k(1+r_c'^2)}{r_c'^2}\Delta - \frac{k}{R(\alpha_1)}\left\{z'(\alpha_1) - \frac{R'(\alpha_1)}{R(\alpha_1)}[z(\alpha_1) - z(\alpha_0)] - \right.$$

$$\left. - \frac{z'(\alpha_0)}{r_c'\cot\alpha_0}\right\} - \left[\left(\frac{1}{r_c'} - \frac{1}{r_1}\right)\Delta + \frac{z(\alpha_1) - z(\alpha_0)}{R(\alpha_1)} - \frac{1}{r_1}\right]\left[k_{\alpha_1}' + \frac{k_{\alpha_0}'}{r_c'\cot\alpha_0}\right]$$

Eliminating the variations from (10) and (12), we have

$$M_1(r_c, r_c')N_2(r_c, r_c') - N_1(r_c, r_c')M_2(r_c, r_c') = 0 \tag{13}$$

Equations (11) and (13), obtained in closed form, permit the determination of the quantities r_c and r_c'. All remaining parameters, including the coordinate c of the juncture point, are then found immediately from (8).

Solution of the equations is significantly simplified if one considers a body depending on only the one parameter r_1, for example by taking $r_0 = 0$. However, in axisymmetric flow it is impossible to obtain a body completely without a duct (as in the theory of Newton), because according to (8) the quantity r_0 cannot be taken equal to zero. Therefore, we require that r_0 be a minimum for a given r_1. Then according to (8) one parameter is given by the condition $\alpha_0 = 1/2\ \pi$. We will regard such a body as practically solid, because even for thick bodies r_0/r_1 is less than 1 per cent. In this case, Equations (11) and (13) lead, after elementary calculation, to a single cubic equation of the form

$$\Delta^3 + 3\Delta^3 + a\Delta + b = 0, \qquad a = 6\frac{k(\alpha_1)/\sin^2\alpha_1 - l}{k_{\alpha_1}'(\alpha_1)/k(\alpha_1) - 2l}$$

$$b = \frac{6k(\alpha_1)}{k_{\alpha_1}'(\alpha_1)/k(\alpha_1) - 2l}\left\{\left[\frac{z(\alpha_1)}{R(\alpha_1)} - \cot\alpha_1\right] l - \frac{z(\alpha_1)}{R(\alpha_1)}\left[\frac{z'(\alpha_1)}{z(\alpha_1)} - \frac{R'(\alpha_1)}{R(\alpha_1)}\right]\right\} \qquad (14)$$

$$l\left(2 + \tan^2\alpha_1 - \tan\alpha_1\frac{z(\alpha_1)}{R(\alpha_1)}\right) = \left\{\frac{k(\alpha_1)}{\sin^2\alpha_1\cos^2\alpha_1} - \tan\alpha_1\frac{z(\alpha_1)}{R(\alpha_1)}\left[\frac{z'(\alpha_1)}{z(\alpha_1)} - \frac{R'(\alpha_1)}{R(\alpha_1)}\right]\right\}$$

The quantities necessary for constructing the optimal curve are found in the following way. Given a value of the parameter α_1, one then calculates the coefficients a, b and l, and determines the quantity Δ with the use of Equation (14). After this the relative thickness r_1 of the body and the coordinate c of the juncture-point are found from (11) and (8) according to the formulas

$$r_1 = \frac{6k(1+\Delta)}{\Delta^3 + 3\Delta^2 + 6k[\Delta\cot\alpha_1 + z(\alpha_1)/R(\alpha_1)]}, \qquad c = \frac{z(\alpha_1)}{R(\alpha_1)}\frac{r_1}{1+\Delta} \qquad (15)$$

Results of calculation for various values of the parameter α_1 are given in the table. For a range of values of r_1 from about 0.1 to 0.7 the coordinate of the juncture-point is found to lie in the interval $0.6 \leqslant c \leqslant 0.7$ (for plane flow $c = 0.5$). In Fig. 2 the solid lines show some optimal curves calculated in the indicated manner from Equations (4) and (8). For comparison, the broken lines show the form of the optimal bodies according to Newton [1], which to the scale of the figure are indistinguishable from a curve proportional to $x^{3/4}$. The dependence of the drag of the optimal body (5) on the thickness ratio is shown in Fig. 3 (curve 1). The corresponding dependence for the body of optimum form

α_1	r_1/r_c	c	C_x	r_1
6°	1.345	0.607	0.011	0.116
8°30	1.335	0.615	0.030	0.170
11°20	1.328	0.620	0.054	0.230
14°	1.322	0.625	0.082	0.290
16°50	1.282	0.644	0.121	0.365
22°	1.248	0.655	0.216	0.532
26°30	1.206	0.704	0.335	0.673
28°50	1.183	0.724	0.401	0.767

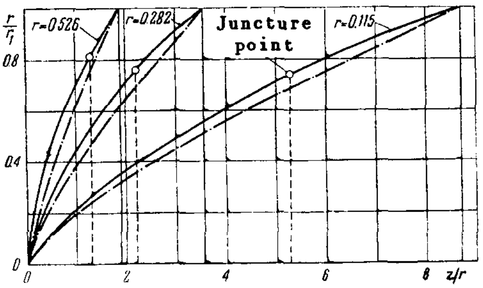

Fig. 2.

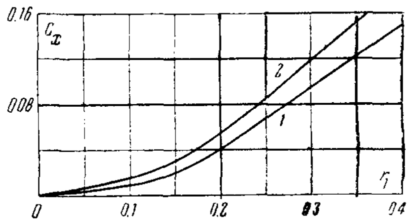

Fig. 3.

according to Newton is also shown (curve 2). Comparison has shown that the drag of the optimal body is less than that of the known optimal body of Cole by 15 to 17 per cent and that of the Newton or $x^{3/4}$ body by 20 to 22 per cent. In some cases, when the thickness of the body is moderate, it is convenient to expand Equation (14) as a series in the parameter a_1.

Then we have

$$a = 3\left(1 - \frac{5}{4}\alpha_1^2\right) + O(\alpha_1^4), \quad b = -\frac{3}{2}\left(1 - \frac{7}{4}\alpha_1^2\right) + O(\alpha_1^4)$$

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